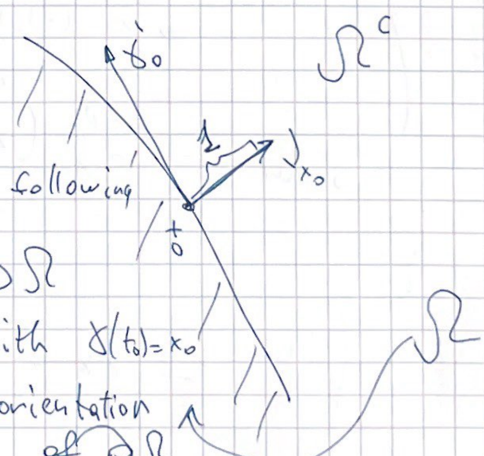
Applications of Green's theorem

1st: Divergence theorem

Need to define ~~outer~~ exterior unit normal.

Let Ω be a regular domain, then the exterior unite normal ν_{x_0} to $\partial\Omega$ at a point $x_0 \in \partial\Omega$ is a vector s.t.:

1. $|\nu_{x_0}| = 1$ (unit condition)
2. for a regular parametrization $\gamma : [a,b] \to \partial\Omega$ s.t. $\gamma(t_0) = x_0$ for some $t_0 \in [a,b]$ we get
   $$\langle \gamma'(x_0), \nu_{x_0} \rangle = 0 \quad \text{(normality condition)}$$
3. $\exists\ \varepsilon_0 > 0$ s.t. $\forall\ 0 < \varepsilon < \varepsilon_0$ we get $x_0 + \varepsilon \nu_{x_0} \notin \Omega$ (Exterior condition)



We can find ν_{x_0} in the following way: Let $\gamma : [a,b] \to \partial\Omega$ be a parametrization with $\gamma(t_0) = x_0$ and inducing positive orientation of $\partial\Omega$

then
$$\nu_{\gamma(t)} = \frac{1}{|\gamma'(t)|} \cdot \left(\gamma_2'(t), -\gamma_1'(t)\right)$$

Divergence theorem in $\mathbb{R}^2$

Let Ω be a regular domain, $F \in C^1(\Omega, \mathbb{R}^2)$ a vector field

Then

$$\iint_{\Omega} \mathrm{div}\left(F(x,y)\right)\, dx\, dy = \int_{\partial\Omega} \langle F, \nu \rangle \, dl$$

I finished lecture with formulating divergence thm

<u>Example</u> Let's verify divergence thm for

$$\Omega = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x^2 < y < 1\} \text{ and}$$

$$F : \Omega \longrightarrow \mathbb{R}^2 \text{ given by } F(x,y) = (xy, x^2).$$

On one side we get:

$$\iint_{\Omega} \mathrm{div}\, F \, dx dy = \int_{-1}^{1}\int_{x^2}^{1} y \, dy dx = \int_{-1}^{1} \frac{1}{2} y^2 \Big]_{x^2}^{1} dx =$$

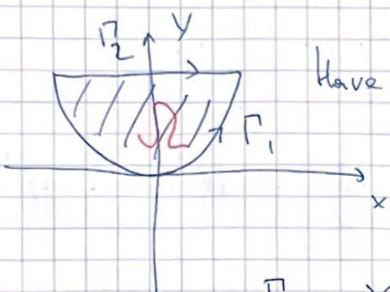
$\hookrightarrow x \in (-1,1)$, $y^2 \in (x^2, 1)$

$$\mathrm{div}\, F = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(x^2) = y$$

$$= \int_{-1}^{1} \frac{1}{2} - \frac{x^4}{2} \, dx = \left[\frac{1}{2}x - \frac{x^5}{10}\right]_{-1}^{1} = \boxed{\frac{4}{5}}$$

On the other hand:

Computing $\int_{\partial\Omega} \langle F, \nu \rangle \, d\ell$. First choose parametrization:



Have two boundary arcs: $\Gamma_1 \cup \Gamma_2 = \Gamma$

parametrizing $\partial\Gamma$ gives

$$\Gamma_1: \quad \gamma_1(t) = (t, t^2) \quad \text{For } t \in [-1, 1]$$

$$\gamma_1'(t) = (1, 2t) \quad \text{and}$$

$$\nu_{\gamma(t)} = \frac{(2t, -1)}{\sqrt{4t^2+1}}$$

$$\Gamma_2: \quad \gamma_2(t) = (t, 1) \qquad \gamma_2'(t) = (1, 0) \Rightarrow \nu_{\gamma_2(t)} = (0, -1)$$

$$\int_{\partial\Omega} \langle F, \nu \rangle \, d\ell = \int_{-1}^{1} \langle F(\gamma_1(t)), \nu_{\gamma_1}(t) \rangle \cdot |\gamma_1'(t)| \, dt - \int_{-1}^{1} \langle F(\gamma_2(t)), \nu_{\gamma_2(t)} \rangle \cdot |\gamma_2'(t)| \, dt =$$

since Γ_2 is oriented in a wrong way

$$= \int_{-1}^{1} \left\langle (t^3, t^2), \frac{(2t, -1)}{\sqrt{4t^2+1}} \right\rangle \cdot \sqrt{4t^2+1} \, dt - \int_{-1}^{1} \langle (t, t^2), (0, -1) \rangle \, dt$$

$$= \int_{-1}^{1} 2t^4 - t^2 + t^2 \, dt = \frac{2}{5} t^5 \Big]_{-1}^{1} = \boxed{\frac{4}{5}}$$

Application 2: <u>Area Formulas:</u>

<u>Theorem:</u> Let $\Omega \subseteq \mathbb{R}^2$ be a regular domain and define F, G, H to be:

$$F(x,y) = (-y,x) \ ; \ G(x,y) = (-y,0), \ H(x,y) = (0,x)$$

Then. $$\text{Area}(\Omega) := \iint_{\Omega} 1 \, dx\,dy = \frac{1}{2} \int_{\partial\Omega} F \cdot dl = \int_{\partial\Omega} G \cdot dl = \int_{\partial\Omega} H \cdot dl$$

<u>Proof</u> $\operatorname{rot}(F) = \frac{\partial}{\partial x} x - \frac{\partial}{\partial y}(-y) = 1 - (-1) = 2$

$\operatorname{rot}(G) = \operatorname{rot}(H) = 1$ and use Green's thm.

<u>Application 3</u> Green's Identities on $\mathbb{R}^2$

<u>Thm</u> Let $\Omega \subset \mathbb{R}^2$ be a regular domain, ν ~~the~~ be the outer unit normal and $u, v \in C^2(\bar{\Omega})$ then we have the following identities:

1. $$\iint_{\Omega} \Delta u \, dx\,dy = \int_{\partial\Omega} \langle \operatorname{grad} u, \nu \rangle \, dl$$
2. $$\iint_{\Omega} v \Delta u + \langle \operatorname{grad} u, \operatorname{grad} v \rangle \, dx\,dy = \int_{\partial\Omega} \langle v \cdot \operatorname{grad} u, \nu \rangle \, dl$$
3. $$\iint_{\Omega} (u \Delta v - v \Delta u) \, dx\,dy = \int_{\partial\Omega} \langle u \cdot \operatorname{grad}(v) - v \cdot \operatorname{grad}(u), \nu \rangle \, dl.$$

Surfaces and surface integrals

Chapter 5

We will denote components of vector field with upper indices:

$$g = (g^1, g^2 \dots g^n) \quad \text{for} \quad g : \mathbb{R}^n \to \mathbb{R}^n$$

We also use the notation:

$$f_x = \frac{\partial f}{\partial x}, \quad f_y = \frac{\partial f}{\partial y}, \quad f_z = \frac{\partial f}{\partial z}$$

for $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$.

Definition $\Sigma \subset \mathbb{R}^3$ is called a regular surface if

1). $\exists\, A \subset \mathbb{R}^2$ bounded s.t. ∂A is a simple closed ~~curve~~ picewise regular curve and:

$\exists\, \sigma : A \longrightarrow \mathbb{R}^3$ with $\sigma(\bar{A}) = \Sigma$;
$\sigma \in C^1(\bar{A}, \mathbb{R}^3)$ and σ is injective on A.

2) In Addition, $\sigma_u \wedge \sigma_v := (\sigma_u^1, \sigma_u^2, \sigma_u^3) \wedge (\sigma_v^1, \sigma_v^2, \sigma_v^3) :=$

$$= \begin{pmatrix} \sigma_u^2 \sigma_v^3 - \sigma_v^2 \sigma_u^3 \\ \sigma_v^1 \sigma_u^3 - \sigma_u^1 \sigma_v^3 \\ \sigma_u^1 \sigma_v^2 - \sigma_v^1 \sigma_u^2 \end{pmatrix} \rightsquigarrow \text{Matrix of minors of } \begin{pmatrix} \sigma_u^1 & \sigma_u^2 & \sigma_u^3 \\ \sigma_v^1 & \sigma_v^2 & \sigma_v^3 \end{pmatrix}$$

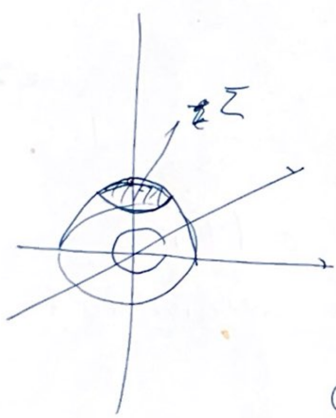
is non-zero for every $x \in A$.

Such σ is called regular parametrization of Σ and $\dfrac{\sigma_u \wedge \sigma_v}{|\sigma_u \wedge \sigma_v|} = \nu_{(u,v)}$ is called a unit normal to a point $\sigma(u,v)$.

A piecewise regular surface

Σ is piecewise regular if $\exists$ k and $\Sigma_1 \ldots \Sigma_k$ s.t. $\Sigma = \bigcup_{i=1}^{k} \Sigma_i$ and $\Sigma_1 \ldots \Sigma_k$ are regular.

Example: take $\Sigma = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1,\ x^2+y^2 \le \frac{1}{2},\ z \geq 0 \}$



We will use cylinder coordinates:

$$(x,y,z) = (r\cos\theta, r\sin\theta, z)$$

Condition $x^2+y^2+z^2=1$ and $z \geq 0$ implies that $z = \sqrt{1-r^2}$, $x^2+y^2 \le \frac{1}{2} \Rightarrow$ $\Rightarrow r \in [0, \frac{1}{\sqrt{2}}]$

So: take $A = (0, \frac{1}{\sqrt{2}}) \times (0, 2\pi)$ $\quad (\bar{A} = [0, \frac{1}{\sqrt{2}}] \times [0, 2\pi])$

And $\sigma : \bar{A} \to \mathbb{R}^3$ $\quad \sigma(r,\theta) = (r\cos\theta, r\sin\theta, \sqrt{1-r^2})$

We compute

$$\sigma_r \wedge \sigma_\theta = \begin{pmatrix} \frac{r^2}{\sqrt{1-r^2}} \cos\theta \\ \frac{r^2}{\sqrt{1-r^2}} \sin\theta \\ r \end{pmatrix}$$ which is never 0

Also $\sigma(\bar{A}) = \Sigma$ so we get a regular parametrization

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<u>Definition</u> Surface is orientable if there exists a ~~a~~ continuous field of normal vectors $\nu : \Sigma \to \mathbb{R}^3$. The data of such field is called ~~A~~ an orientation

Remark: might have problem at gluings: map $\sigma : \bar{A} \to \mathbb{R}^3$ is not injective on ∂A:

<u>Definition</u> Let $\Omega \subset \mathbb{R}^3$ be open, $f \in C^0(\Omega)$, $F \in C^0(\Omega, \mathbb{R}^3)$

and $\Sigma \subset \Omega$ be a regular surface with parametrization $\sigma : \overline{A} \to \Sigma$

then we define:

1 Integral of f:

$$\iint_\Sigma f \, ds := \iint_A f(\sigma(u,v)) \cdot |\sigma_u(u,v) \wedge \sigma_v(u,v)| \, du \, dv$$

2. Integral of F:

$$\iint_\Sigma F \cdot ds := \iint_A \left\langle F(\sigma(u,v)), \; \sigma_u(u,v) \wedge \sigma_v(u,v) \right\rangle du \, dv$$

As usual for piecewise regular surfaces

$$\iint_\Sigma f ds = \sum_{i=1}^{k} \iint_{\Sigma_i} f ds \; ; \quad \iint_\Sigma F \cdot ds = \sum_{i=1}^{k} \iint_{\Sigma_i} F \cdot ds.$$